[1] For formulas cited without proof see W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea, 1949).

[2] This theorem imitates known facts for Fourier expansions on compact group and homogeneous spaces. For summability at points see G. Szegö, *Orthogonal Polynomials* (1939) and references there listed to publications by Kogbetliantz, Gronwall, Fejér, and others.

[3] To Theorems 3 and 4 compare S. Bochner, "Closure Classes Originating in the Theory of Probability," these PROCEEDINGS, **39**, 1082–1088, 1953.

[4] See S. Bochner, "Diffusion Equation and Stochastic Processes," these PROCEEDINGS, **35**, 368–370, 1949.

KÄHLERIAN COSET SPACES OF SEMISIMPLE LIE GROUPS

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Our main purpose in this note is to determine the coset spaces of semisimple Lie groups which admit a complex analytic Kählerian structure invariant under the group; we shall also obtain some information on coset spaces with an invariant symplectic structure. In the compact case, all the manifolds thus obtained are algebraic and admit a complex analytic cellular decomposition; in the noncompact case, they are complex analytically fibered, with compact Kählerian fibers, over Hermitian symmetric spaces. As an application, we see that a bounded domain in the space of several complex variables which has a transitive *semisimple* group of complex analytic homeomorphisms is symmetric in É. Cartan's sense,[1] thus giving a *partial* answer to a well-known question raised by that author. Only brief indications of proofs are given; the full details will appear elsewhere.

1. *Notations and Definitions.*—G denotes a connected Lie group, which, except in section 1, *is always supposed to be semisimple;* U is a closed subgroup of G; and G/U is the space of left cosets of G modulo U, on which G acts by the left translations. We always assume G to be effective on G/U, i.e., U contains no subgroup $\neq \{e\}$ invariant in G. Lie algebras are denoted by German letters and, unless otherwise stated, are taken over the real numbers; the Lie algebra of a group G, $U, \ldots$ is of course denoted by the corresponding German letter.

A complex analytic manifold is Kählerian if it is endowed with a Hermitian metric whose imaginary part Ω, the so-called associated form to the metric, has exterior differential zero; in any case it is an exterior form of degree two and maximal rank everywhere. An even-dimensional manifold carrying a form Ω with the last-named properties is called *symplectic;* it is always orientable; clearly, Kählerian implies symplectic.

A coset space is homogeneous complex (resp. homogeneous Kählerian, resp. homogeneous symplectic) if it carries a complex analytic structure (resp. Kählerian structure, resp. a form of degree two and maximal rank everywhere) invariant under the group. For a compact connected group the usual averaging process shows that symplectic implies homogeneous symplectic and that "homogeneous complex and Kählerian" implies "homogeneous Kählerian."

$H^i(X)$ (resp. $H^i(X, Z)$) is the ith cohomology group of the manifold X with real coefficients (resp. integers).

2. *A Necessary Condition.*—A Lie algebra is reductive[2] if its adjoint representation is fully reducible or, equivalently,[3] if it is the direct product of its center by a semisimple ideal; a Lie subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{a}$ is reductive in $\mathfrak{a}$ if the restriction to $\mathfrak{b}$ of the adjoint representation of $\mathfrak{a}$ is fully reducible in $\mathfrak{a}$.

PROPOSITION 1. *Let G/U be homogeneous symplectic, U be connected. Assume $\mathfrak{u}$ to be reductive in $\mathfrak{g}$, and let $\mathfrak{c}$ be its center. Then $\mathfrak{u}$ is the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$.*

The proof makes mainly use of cohomology of Lie algebras and is based on the three following facts: (*a*) a semisimple Lie algebra has vanishing first and second cohomology groups; (*b*) the centralizer of $\mathfrak{c}$ in $\mathfrak{g}$ is reductive in $\mathfrak{g}$; (*c*) there exists an element h in the second relative cohomology group $H^2(\mathfrak{g}, \mathfrak{u})$ such that $h^m \neq 0$ ($2m = \dim G/U$).

Remarks: (1) It follows from Proposition 1 that $\mathfrak{u}$ contains a Cartan subalgebra of $\mathfrak{g}$. (2) Proposition 1 applies when G/U is homogeneous Kählerian, because in that case $\mathfrak{u}$ is readily seen to be reductive in $\mathfrak{g}$, even when U is not compact. It also applies for G compact, G/U symplectic. (3) For a particular case of Proposition 1, see A. Lichnerowicz, *Compt. Rend. Acad. Sci.* (*Paris*) **237**, 695–697 (1953).

THEOREM 1. *Let either G/U be homogeneous Kahlerian, or G be compact and G/U be symplectic. Then U is compact, connected, and equal to the centralizer of a torus of G. Moreover, G has center reduced to $\{e\}$,[4] and G/U is simply connected.*

For U connected, this follows essentially from Proposition 1 and remark 1. In the general case, one considers the covering G/U_0 (U_0 connected component of the identity in U), on which the given structure on G/U induces a similar structure, invariant under G, and also under U/U_0 operating by right translations; the main point is that U/U_0 operates faithfully on the identity component of the center of U_0, and that follows from the lemma: If the centralizer of a torus in a connected semisimple Lie group has a compact identity component, then it is equal to it.

COROLLARY. *Let $G = G_1 \times \ldots \times G_k$ be a decomposition of G into a product of simple groups. Then $U = U_1 \times \ldots \times U_k$ where $U_i \subset G_i$ and is the centralizer of a torus in G_i; hence G/U is isomorphic[5] to the product of the spaces G_i/U_i.*

3. *Complex Semisimple Lie Algebras.*—We recall here a few known facts and fix some notations. Let $\mathfrak{g}$ be a compact semisimple Lie algebra of rank l, dimension $n = l + 2m$; let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and $\mathfrak{h}^c$ be a Cartan subalgebra of $\mathfrak{g}^c$ such that $\mathfrak{h} = \mathfrak{h}^c \cap \mathfrak{g}$ has (real) dimension l. We denote by $\pm 2\pi i a_j$ ($1 \leqq j \leqq m$) the roots of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$; the a_j's are therefore real-valued on $\mathfrak{h}$, and, moreover, we assume them to be positive with respect to some total ordering of the space $\mathfrak{h}^*$ dual to $\mathfrak{h}$, chosen once for all, the fundamental roots being $2\pi i a_k$ ($1 \leqq k \leqq l$). The scalar product induced on $\mathfrak{h}$ or $\mathfrak{h}^*$ by the Killing form is written (,), and W is the Weyl chamber (in $\mathfrak{h}$ or $\mathfrak{h}^*$) defined by $(a_k, y) \geqq 0$ ($1 \leqq k \leqq l$).

$e_{\epsilon j}$ ($\epsilon = \pm 1, j = 1, \ldots, m$) is an element of $\mathfrak{g}^c$ satisfying

$$[h, e_{\epsilon j}] = \epsilon 2\pi i a_j(h) \qquad (h \in \mathfrak{h}),$$

chosen in the usual way, and $\mathfrak{g}$ is spanned over the reals by $\mathfrak{h}$, by $e_j + e_{-j}$ and $i(e_j - e_{-j})$ ($1 \leqq j \leqq m$).

Let G^c (resp. G) be the group with center reduced to $\{e\}$ and Lie algebra $\mathfrak{g}^c$ (resp. $\mathfrak{g}$). We denote by L the closed solvable subgroup of G^c generated by $\mathfrak{h}^c$ and the e_j's $(1 \leqq j \leqq m)$ and by L_b $(b \in W)$ the subgroup generated by $\mathfrak{h}^c$, the e_j's $(1 \leqq j \leqq m)$, and the e_{-k} for which $(a_k, b) = 0$ (it is readily seen that these elements form the Lie algebra of a closed subgroup).

$\mathfrak{h}^*$ may be identified in a well-known way to $H^1(T)$, where T is a maximal torus of G with Lie algebra $\mathfrak{h}$; in this identification $H^1(T, Z)$ becomes the set of $h \in \mathfrak{h}^*$ for which $2(a_k, h)(a_k, a_k)^{-1}$ $(1 \leqq k \leqq l)$ is an integer.

4. *The Compact Case.*—If G is compact semisimple, then G/U has the first Betti number zero; conversely, any compact coset space of a Lie group with vanishing first Betti number is a quotient of a compact group[6] which, as is easily seen, may be assumed to be semisimple. Hence the results of sections 2 and 4 give *all algebraic homogeneous manifolds with first Betti number zero.*

THEOREM 2. *Let G be compact semisimple and U be the centralizer of a torus. Then G/U is homogeneous Kählerian and algebraic.*[7]

There exists clearly $b \in W$ such that $\mathfrak{u}$ is the centralizer of b; one proves that $U = G \cap L_b$ (notations of sec. 3), whence a natural homeomorphism of G/U onto G^c/L_b commuting with G and the homogeneous complex structure. The invariant Kählerian metric is then constructed by means of Maurer-Cartan forms. We sketch here the proof for G/T (T maximal torus), i.e., $L_b = L$; the general case is analogous.

Let us denote by $\omega^{\epsilon j}$ the left-invariant Maurer-Cartan forms on G^c which induce on $\mathfrak{g}^c$ the base dual to $(e_{\epsilon j})$ and are orthogonal to $\mathfrak{h}^c$; using the Maurer-Cartan equations and well-known properties of constants of structure, one shows that

$$\Omega = i \sum_{= 1}^{j = m} c_j \omega^j \wedge \omega^{-j}$$

is closed if and only if

$$c_p + c_q = c_r \qquad \text{whenever} \qquad a_p + a_q = a_r.$$

Ω is therefore determined by the c_k's $(1 \leqq k \leqq l)$, which are arbitrary; its restriction on G is left-invariant under G, right-invariant under T, and represents a form on G/T, which is of type $(1, 1)$ because ω^{-j} corresponds to $\bar{\omega}^j$ in the complex structure constructed above. For real c_j's it is real-valued, and its (real) cohomology class may be shown to be the image by transgression of the element $h \in H^1(T)$, for which $(a_k, h) = c_k$ $(1 \leqq k \leqq l)$. If h belongs to the interior of W, all the c_j's are > 0, and

$$ds^2 = \sum_{j = 1}^{j = m} c_j \omega^j \cdot \bar{\omega}^j \qquad \text{(usual product)}$$

is a Kählerian metric on G/T. If, moreover, $h \in H^1(T, Z)$, its image by transgression is an integral class, the corresponding metric is a Hodge metric, and G/T is algebraic by a result of Kodaira.[8]

Remark: This theorem can be proved in other ways; for instance, one can construct projective imbeddings with the help of linear representations, as was noticed by J. Tits and, independently, by A. Weil and the author (yet unpublished); also, M. Goto proved that G/U is rational algebraic (to appear in *Am. J. Math*); finally, the existence of the homogeneous complex structure is part of a result of H. C. Wang (*Am. J. Math.*, **76**, 1–32, 1954). The above method has been sketched, however, because it is also used in section 6.

5. *Complex Analytic Cellular Decompositions.*—A complex compact manifold M has a complex analytic cellular decomposition if it admits a partition into a finite number of (complex) submanifolds M_i, the "open cells," each isomorphic to some complex affine space and having a set-theoretical boundary made up of open cells with strictly smaller dimensions; the closures of the M_i's define, then, a cellular decomposition, whose cells have *even* dimensions; consequently, they are all cycles, and they form a basis for the integral homology groups of M, which thus have no torsion and vanish in odd dimensions. As is well known, C. Ehresmann[9] proved the existence of such decompositions for certain classical spaces, like the complex Grassmann manifolds or the nondegenerate complex quadrics; his method is geometric and uses mainly the Schubert systems, but this result can also be given a group-theoretical proof, valid for all spaces considered in section 4.

THEOREM 3. *Let G be compact and G/U be homogeneous Kählerian. Then G/U admits a complex analytic cellular decomposition by "open cells" which are birationally and biregularly equivalent to complex affine spaces; in particular, its integral homology groups have no torsion.*[10]

If we represent G/U in the form G^c/L_b as above, the open cells are just the orbits of L; to prove it, one uses notably a recent (unpublished) result of Harish-Chandra (first checked by Bruhat for the classical groups, as announced in *Compt. rend. Acad. sci.* (Paris), **238**, 437, 1954), to the effect that the double cosets LgL of L in G^c are finite in number.

6. *The Noncompact Case.*—A Hermitian manifold is *Hermitian symmetric* if every point is an isolated fixed point of an involutive automorphism of the Hermitian structure; it is then always homogeneous Kählerian.[11] The quotient G/K of a simple noncompact group with center reduced to $\{e\}$ by a maximal compact subgroup carries a Hermitian symmetric structure invariant under G if and only if K has a nondiscrete center.[11]

PROPOSITION 2. *Let G be simple noncompact, with center reduced to $\{e\}$, K a maximal compact subgroup of G, and U a subgroup of K which is the centralizer in G of a torus. Then G/U is homogeneous complex and homogeneous symplectic.*[12] *It is homogeneous Kählerian if and only if G/K is Hermitian symmetric; in that case, the fibering of G/U by K/U over G/K is complex analytic.*

Let G_c be a maximal compact subgroup of the complexification G^c of G, containing K; the group U is also centralizer of a torus in G_c, and (Theorem 2) G_c/U is homogeneous Kählerian; it is easily shown that G/U may be identified with an open submanifold of G_c/U, whence the homogeneous complex structure. The second part of the theorem is obtained by detailed analysis of invariant differential forms. From that and from the corollary to Theorem 1, one gets Theorem 4.

THEOREM 4. *The homogeneous Kählerian coset spaces of semisimple Lie groups are all simply connected. They are exactly the products of the Kählerian homogeneous spaces G_i/U_i with G_i simple, U_i centralizer of a torus, and where either G_i is compact or G_i has a maximal compact subgroup with nondiscrete center. Any coset space G/U of this type has a complex analytic fibering with fiber K/U (K maximal compact) over a Hermitian symmetric space.*

A bounded domain in C^n possesses a Kählerian metric invariant under all complex analytic homeomorphisms (the Bergmann metric); hence, if it is homogeneous, it is automatically homogeneous Kählerian. It is said to be *symmetric*[1] if every point is an isolated fixed point of an involutive complex analytic homeomorphism; this implies Kählerian homogeneity. É. Cartan[1] has asked whether every bounded homogeneous domain in C^n is symmetric and has checked that it is indeed the case for $n = 1, 2, 3$. Since a domain does not contain a connected compact complex analytic submanifold with more than one point, we deduce from Theorem 4:

THEOREM 5. *A bounded domain in C^n which admits a transitive semisimple group of complex analytic homeomorphisms is symmetric.*[13]

[1] É. Cartan, *Abhandl. Math. Sem. Hamburg*, **11**, 116–162, 1935.

[2] J. L. Koszul, *Bull. Soc. Math. France*, **78**, 65–127, 1950.

[3] *Ibid.*, p. 87.

[4] Recall that G is effective on G/U by assumption.

[5] As coset space only.

[6] D. Montgomery, *Proc. Am. Math. Soc.*, **1**, 467–469, 1950.

[7] I.e., is complex analytically homeomorphic to a complex submanifold of some complex projective space, imbedded without singularities.

[8] K. Kodaira, these PROCEEDINGS, **40**, 313–316, 1954.

[9] C. Ehresmann, *Ann. Math.*, **35**, 396–443, 1934.

[10] This applies, e.g., to G/T (T maximal torus), as had been partly checked by the author (*Ann. Math.*, **57**, 115–207, 1953, sec. 29). The absence of torsion on these spaces has also been proved by R. Bott, these PROCEEDINGS, **40**, 586–588(1954).

[11] A. Borel and A. Lichnerowicz, *Compt. rend. Acad. sci.* (Paris), **234**, 2332–2334, 1952.

[12] In fact, it always carries an invariant *indefinite* Kählerian metric.

[13] Theorem 5 has also been obtained independently by J. L. Koszul (yet unpublished). For the homogeneous spaces of theorem 2, see also J. Tits, *Compt. Rend. Acad. Sci. (Paris)*, **239**, 466–468(1954).

THE GREEN'S AND NEUMANN'S PROBLEMS FOR DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS

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1. *Definition of the Green's and Neumann's Spaces.*—The classical Green's and Neumann's problems for a function on a subdomain of Euclidean n-space are the problems of finding a harmonic function with prescribed values on the boundary for the function and its normal derivative respectively. Our purpose here is to state a generalization of these problems to differential forms on an arbitrary Riemannian manifold; detailed proofs will appear elsewhere.

By "differentiable" we shall always mean "differentiable of class C^∞," and by "manifold" we shall mean "orientable differentiable manifold." All Riemannian structures are therefore of class C^∞.

On a Riemannian manifold of dimension n we have, in addition to the operator d of exterior differentiation, its formal adjoint $\delta = (-1)^{np+n+1}*d*$ where $*$ denotes the usual duality operator carrying a differential form of degree p (p-form) into one